ABSOLUTE RETRACTS AS REDUCED PRODUCTS

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ABSTRACT. We show that in finitely generated congruence distributive varieties every absolute retract is a product of reduced powers of maximal subdirectly irreducibles.

In [OR96] it was shown that if an absolute retract in a finitely generated congruence distributive variety has a subdirect representation in which *none of the factors are proper homomorphic images of each other*, then such an absolute retract is a product of reduced powers of maximal subdirectly irreducibles. Here we show that the same conclusion can be reached for any absolute retract in a finitely generated congruence distributive variety.

Let $f : \mathbf{A} \hookrightarrow \prod_{j\in J} \mathbf{B}_j$ be a subdirect embedding, and let us call f *weakly irredundant* if it satisfies the following condition:

for all $k \in J$ there exist $x, y \in A$ such that $\pi_k f(x) \neq \pi_k f(y)$,
and for all $j \in J$, whenever $\pi_j f(x) \neq \pi_j f(y)$ then $B_j \cong B_k$.

Recall that a subdirect embedding is *irredundant* if none of the factors can be omitted, i.e. for all $k \in J$ there exist $x, y \in A$ such that for all $j \in J$, whenever $\pi_j f(x) \neq \pi_j f(y)$ then $j = k$. Hence any irredundant subdirect embedding is weakly irredundant. However, irredundant subdirect embeddings need not exist, even in finitely generated varieties (e.g. any subdirect representation of an atomless Boolean algebra will remain a representation if finitely many factors are omitted). Fortunately weakly irredundant subdirect embeddings exist and are sufficient for our purpose.

Lemma 1. *Let $f : \mathbf{A} \hookrightarrow \mathbf{B} = \prod_{j\in J} \mathbf{B}_j$ be a weakly irredundant subdirect embedding in a congruence distributive variety, and suppose all factors are finite subdirectly irreducibles. If $r : \mathbf{B} \twoheadrightarrow \mathbf{A}$ is a retraction, then* $\ker r$ *is a filtral congruence.*

Date: March 4, 1999.

1991 *Mathematics Subject Classification.* Primary .

Key words and phrases. absolute retracts, reduced products, maximal irreducibles.

The second and fourth author's work was supported by grants from the South African Foundation for Research Development, and the University of Cape Town Research Committee.

Proof. For each $k \in J$, consider the composite homomorphism $h_k = \pi_k \circ f \circ r : \mathbf{B} \twoheadrightarrow \mathbf{B}_k$. By Jónsson's Lemma there exists an ultrafilter U on J such that the congruence $\phi_U \in \mathrm{Con}(\mathbf{B})$ induced by U is a subset of $\ker h_k$. By weak irredundance we can choose $x, y \in A$ such that $\pi_k f(x) \neq \pi_k f(y)$ and for all $j \in J$, $\pi_j f(x) \neq \pi_j f(y)$ implies $\mathbf{B}_j \cong \mathbf{B}_k$.

Since r is a retraction, $h_k \circ f = \pi_k \circ f \circ r \circ f = \pi_k \circ f$, so $h_k f(x) \neq h_k f(y)$. It follows that the pair $(f(x), f(y))$ is not in the kernel of h_k, hence also not in ϕ_U. This means $\{j \in J : \pi_j f(x) \neq \pi_j f(y)\} \in U$, and since $\pi_j f(x) \neq \pi_j f(y)$ implies $\mathbf{B}_j \cong \mathbf{B}_k$, we obtain $\{j \in J : \mathbf{B}_j \cong B_k\} \in U$. But $\mathbf{B}_k$ was assumed to be finite, whence $\mathbf{B}/\phi_U \cong B_k \cong \mathbf{B}/\ker h_k$, and therefore $\phi_U = \ker h_k$. Thus each h_k is induced by an ultrafilter.

The assumption that f is a subdirect embedding implies that $\ker r = \bigcap \ker h_k$, and since the intersection of filtral congruences is again filtral, the result follows. □

We now show that in a finitely generated congruence distributive variety, every subdirect embedding into a product of subdirectly irreducibles can be reduced to a weakly irredundant one.

Lemma 2. *Let $f : \mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{B}_i$ be a subdirect embedding in a finitely generated congruence distributive variety, and suppose all factors are subdirectly irreducibles. Then there exists a subset J of I such that $\pi_J \circ f : \mathbf{A} \hookrightarrow \prod_{j \in J} \mathbf{B}_j$ is a weakly irredundant subdirect embedding, where π_J is the projection onto the coordinates in J.*

Proof. Define the equivalence relation $\equiv$ on I by $i \equiv j$ iff $\mathbf{B}_i \cong \mathbf{B}_j$. Since we are working in a finitely generated variety, there are only finitely many equivalence classes $I_1, \ldots, I_n$. For $m = 1, \ldots, n$ let

$$\begin{aligned} J_m = \{ i \in I_m :\ & \text{for some } x, y \in A,\ \pi_i f(x) \neq \pi_i f(y) \text{ and} \\ & \text{for all } j \in J_1 \cup \cdots \cup J_{m-1} \cup I_{m+1} \cup \cdots \cup I_n,\ \pi_j f(x) = \pi_j f(y) \}. \end{aligned}$$

We claim that $J = J_1 \cup \cdots \cup J_n$ provides the required weakly irredundant subdirect embedding. First we show that $\pi_J \circ f$ is an embedding. Let x, y be distinct elements of $\mathbf{A}$. Since f is an embedding, there exist $m \in \{1, \ldots, n\}$ and $i \in I_m$ such that $\pi_i f(x) \neq \pi_i f(y)$. We may assume that m is the largest such index.

If $i \in J_m$, then of course $\pi_J f(x) \neq \pi_J f(y)$. If $i \notin J_m$, then for all $x', y' \in A$, either $\pi_j f(x') = \pi_j f(y')$ or there exists $j \in J_1 \cup \cdots \cup J_{m-1} \cup I_{m+1} \cup \cdots \cup I_n$ such that $\pi_j f(x') \neq \pi_j f(y')$. Taking $x' = x$, $y' = y$, and considering the maximality of m, this case reduces to $j \in J_1 \cup \cdots \cup J_{m-1}$ such that $\pi_j f(x) \neq \pi_j f(y)$. So again we have $\pi_J f(x) \neq \pi_J f(y)$.

The weak irredundance of this subdirect decomposition is almost immediate from the definition of the J_m. □

Theorem 3. *In finitely generated congruence distributive varieties every absolute retract is a product of reduced powers of maximal subdirectly irreducibles.*

Proof. Let **A** be an absolute retract in such a variety. By Birkhoff, **A** has a subdirect representation with finite subdirectly irreducible factors, and by Lemma 2 this can be reduced to a weakly irredundant subdirect representation. Since we are assuming **A** is an absolute retract, there exists a retraction back onto **A**, and by Lemma 1 this retraction is filtral. Hence **A** is a reduced product of subdirectly irreducibles.

By standard results on reduced products (see e.g. [OR96] for details) a reduced product with finitely many non-isomorphic factors is isomorphic to a product of reduced powers, and if a reduced power of a finite algebra is an absolute retract, then the algebra itself is an absolute retract. In a finitely generated congruence distributive variety any subdirectly irreducible absolute retract is a maximal subdirectly irreducible, hence the result follows. □

As a corollary we can deduce the special case, (proved in [OR96], and essentially in [Jón90]) that every finite absolute retract in a finitely generated congruence distributive variety is a product of maximal subdirectly irreducibles.

Moreover, in [OR96] it was shown that in a finitely generated congruence distributive variety in which each algebra has a one-element subalgebra (e.g. finitely generated lattice varieties), every equationally compact product of reduced powers of maximal subdirectly irreducibles is an absolute retract. Thus we can now state an internal characterisation of absolute retracts in such varieties, followed by one of its applications.

Corollary 4. *Let $\mathcal{V}$ be a finitely generated congruence distributive variety in which every member has a one-element subalgebra.*

(i) *The absolute retracts of $\mathcal{V}$ are precisely the equationally compact products of reduced powers of maximal subdirectly irreducibles.*
(ii) *If $\mathcal{W}$ is a subvariety of $\mathcal{V}$, and if every maximal subdirectly irreducible algebra in $\mathcal{W}$ is also maximal subdirectly irreducible in $\mathcal{V}$, then every absolute retract of $\mathcal{W}$ is an absolute retract of $\mathcal{V}$.*

By email we have received communication from Keith Kearnes about another characterisation of absolute retracts in fintely generated congruence distributive varieties, in terms of products of Boolean powers. The connections between his and our characterisation are still to be explored.

References

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